§ 3.2: (cont.)

Ex: (§3.2, Problem 38) Find the equation for the tangent line of $y = e^{-2t}$ at $t=0$.

Sol: $\frac{d}{dt}\left(e^{-2t}\right) \underset{\text{(chain rule)}}{=} \frac{d}{dt} f(t)\Big|_{t=g(t)} \cdot \frac{d}{dt} g(t) = \frac{d}{dt} e^{t}\Big|_{t=-2t} \cdot \frac{d}{dt}(-2t)$

$= e^{t}\Big|_{t=-2t} \cdot (-2) = -2e^{-2t}$

$m = \frac{d}{d}\left(e^{-2t}\right)\Big|_{t=0} = -2e^{-2\cdot 0} = -2$

$(0, e^{-2\cdot 0}) = (0,1)$ in on the tangent line

$y = -2x+1$ in tangent line

Ex: (§3.2, Problem 37) Find the equation for the tangent line of $y = 3^{x}$ at $x=1$.

Sol:

§3.3: Chain rule

Obj: We study how to compute derivatives of a compositions of functions.

Thm: (Chain rule) If $f(x)$ and $g(x)$ are functions, then

$$\frac{d}{dx}\left(f(g(x))\right) = \boxed{\left(\frac{d}{dx} f\right)\Big|_{x=g(x)}} \cdot \frac{d}{dx} g(x)$$

[$f\big|_{x=p} = f(p)$]

$$= f'(g(x)) \cdot g'(x)$$

Rmk: The derivative of $f(x)$ evaluated at $g(x)$, multiplied by the devative of $g(x)$.

Ex: If $f(x) = x^2$ and $g(x) = x^2+1$, then compute $\frac{d}{dx}(f(g(x)))$ and $\frac{d}{dx}(g(f(x)))$.

Sol:

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx} f\Big|_{x=g(x)} \cdot \frac{d}{dx} g \qquad \text{(chain rule)}$$

$$= 2x\Big|_{x=g(x)} \cdot \frac{d}{dx}(x^2+1)$$

[Try $\frac{d}{dx}(g(f(x)))$ on your own]

$$= 2 \cdot g(x) \left(\frac{d}{dx}(x^2) + \cancel{\frac{d}{dx}(1)} \right)$$

$$= 2 \cdot g(x) \cdot 2x$$
$$= 4x(x^2+1)$$

Q: Compute $\frac{d}{dx}\left((4x^2+1)^7\right)$ and $\frac{d}{dx}(5^{2\cdot x})$.

Sol:

$$\frac{d}{dx}\left((4x^2+1)^7\right) = \frac{d}{dx}(x^7)\Big|_{x=4x^2+1} \cdot \frac{d}{dx}(4x^2+1)$$

$[f(x) = x^7$
$g(x) = 4x^2+1]$

$$= \boxed{7x^6\Big|_{x=4x^2+1}} \cdot \boxed{\left(\frac{d}{dx}(4x^2) + \cancel{\frac{d}{dx}(1)}\right)}$$

0

$$= \boxed{7\cdot(4x^2+1)^6} \cdot \boxed{8x}$$

$$= 56x \cdot (4x^2+1)^6$$

Rule: If $z = g(t)$ for a function $g(t)$, then

- $\frac{d}{dt}(z^n) = n z^{n-1} \frac{dz}{dt} = n g(t)^{n-1} \frac{dg}{dt}$
- $\frac{d}{dt}(e^z) = e^z \frac{dz}{dt} = e^{g(t)} \cdot \frac{dg}{dt}$
- $\frac{d}{dt}(\ln z) = \frac{1}{z} \frac{dz}{dt} = \frac{1}{g(t)} \cdot \frac{dg}{dt}$

Q: Differentiate the following:

- $(3t^3-1)^5$
- $\ln(q^2+1)$
- e^{-x^2}

Sol: • $\frac{d}{dt}\left(\boxed{(3t^3-1)}^5\right) = \boxed{5(3t^3-1)^4} \cdot \frac{d}{dt}(3t^3-1)$

$= 5(3t^3-1)^4 \cdot 3 \cdot 3 \cdot t^2$

$= 45 \cdot t^2 \cdot (3t^3-1)^4$

• $\frac{d}{dq}\left(\ln(q^2+1)\right) = \frac{d}{dq}\left(f(q)\right)\Big|_{q=g(q)} \cdot \frac{d}{dq} g(q)$

"outside" $f(q) = \ln(q)$ $\qquad = f'(g(q)) \cdot g'(q)$

"inside" $g(q) = q^2+1$ $\qquad = f'(q^2+1) \cdot 2q$

$\frac{d}{dq} f = \frac{1}{q}$ $\qquad = \frac{1}{q^2+1} \cdot 2q$

$\frac{d}{dq} g = \frac{d}{dq}(q^2+1) = 2q$ $\qquad = \frac{2q}{q^2+1}$

Rmk: A useful link for some of the proofs covered so far in the following:

https://tutorial.math.lamar.edu/classes/calci/DerivativeProofs.aspx

Rmk: In §2.3, we discussed relative rate of change of a function $z = f(t)$,

$$\text{"rel. rate of change"} = \frac{f'(t)}{f(t)} = \frac{1}{z} \cdot \frac{dz}{dt}.$$

Note that $\frac{d}{dt}(\ln z) = \frac{1}{z} \cdot \frac{dz}{dt}$. Putting this together, we have

$$\text{"rel. rate of change"} = \frac{d}{dt}(\ln f(t)).$$

Ex: Compute the relative rate of change for the function $z = P_0 e^{kt}$ (note $P_0, k \in (-\infty, \infty)$).

Sol: * On your own *

Ex: Let $f(x) = (2x-3)^3$. Find the equation of the tangent line at

- $x = 0$
- $x = 2$

Sol: * On your own *